RAYLEIGH-TAYLOR INSTABILITY WITH FINITE SKIN DEPTH

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Abstract.
In this work, the Rayleigh-Taylor instability is addressed in a viscous-resistive current slab, by assuming a finite electron skin depth. The formulation is developed on the basis of an extended form of Ohm's law, which includes a term proportional to the explicit time derivative of the current density. In the neighborhood of the rational surface, a viscous-resistive boundary-layer is defined in terms of a resistive and a viscous boundary-layers. As expected, when viscous effects are negligible, it is shown that the viscous-resistive boundary-layer is given by the resistive boundary-layer. However, when viscous effects become important, it is found that the viscous-resistive boundary-layer is given by the geometric mean of the resistive and viscous boundary-layers. Scaling laws of the time growth rate of the Rayleigh-Taylor instability with the plasma resistivity, fluid viscosity, and electron number density are discussed.

Keywords: resistive instabilities, inertial effects, boundary-layers, scaling laws.

1. Introduction

The classical Rayleigh-Taylor instability [1, 2] occurs at the interface of two fluids, when the more dense fluid is supported by the less dense fluid against gravity. In magnetically confined plasmas, the role of the more dense fluid is played by the plasma itself, that of the less dense "fluid", by the magnetic field, and "gravity" is interpreted as the centrifugal acceleration, experienced by the guiding-centers of charged species, following curved magnetic field lines. Actually, the particles are subjected to the centrifugal force [3]

$$\vec{F} = -m_\mathrm{s} v_\mathrm{G}^2 \left(\vec{b} \cdot \nabla \right) \vec{b}, \qquad (1)$$

where m_s is the species mass, v_G, the guiding-center speed, and $\vec{b}$, the unit vector along the curved magnetic field line.

The simplest example considers a circular line of radius r_0. By adopting plane polar coordinates (r, θ), we can put $\vec{b} = \hat{\theta}$ and $\vec{v}_\mathrm{G} = \hat{\theta} v_\mathrm{G}$. Since r_0 is constant, we have $\nabla = \hat{\theta} \partial / (r_0 \partial \theta)$, and given that $\hat{\theta} \cdot \hat{\theta} = 1$ and $\partial \hat{\theta} / \partial \theta = -\hat{r}$, we obtain the more usual form of the centrifugal force,

$$\vec{F} = \hat{r} m_\mathrm{s} \frac{v_\mathrm{G}^2}{r_0}. \qquad (2)$$

The gravitational field is defined as a centrifugal acceleration, by taking the average of Eq. (1) over many gyro-periods,

$$\vec{g} = -v_\mathrm{T}^2 \left(\vec{b} \cdot \nabla \right) \vec{b}, \qquad (3)$$

where $v_\mathrm{T} = \sqrt{<v_\mathrm{G}^2>}$ is the thermal speed, with $< \cdot >$ denoting the above mentioned average. As a matter of fact, Eq. (3) describes the average effect of the curvature of magnetic field lines on a fluid model for a magnetically confined plasma.

2. Current slab

The simplest model for a magnetically confined plasma is the current slab [4, 5]. By adopting Cartesian coordinates (x, y, z), a constant magnetic field B_{0z} is externally applied to an infinite plasma and a current density $j_{0z}(x)$ flows in a plane slab of x-thickness a. According to Ampère's law, a magnetic field $B_{0y}(x)$ is produced and the average effect of the shear of the resulting field lines on the slab is described by a constant gravitational field g_x.

We consider that any field may be decomposed as

$$\phi(\vec{r}, t) = \phi_0(x) + \phi_1(x) \exp\left(\gamma t + \imath \vec{k} \cdot \vec{r} \right), \qquad (4)$$

where ϕ_0 and ϕ_1 are the equilibrium and perturbative fields, respectively, and γ and $\vec{k}$ $(\hat{x} \cdot \vec{k} = 0)$ denote the time growth rate of the field amplitudes and the perturbative wave vector, respectively. Specifically, we suppose a static state of equilibrium for the system (the equilibrium flow field $\vec{v}_0 = 0$) and an incompressible perturbation (as shown in Ref. [6], the condition $\nabla \cdot \vec{v}_1 = 0$, for the incompressibility of the perturbative flow field, replaces the adoption of an adiabatic, or isothermal, equation of state). Particularly, we assume only infinitesimal perturbations $(\mid \phi_1 \mid \ll \mid \phi_0 \mid)$, in the linear approximation (terms of $\mathcal{O}\left(\phi_1^2\right) \sim 0$).

3. Rational surface

If we consider an inviscid, perfectly conducting plasma, with flow field $\vec{v}$, the magnetic flux becomes "frozen" inside the fluid and the electric field in a co-moving frame vanishes, [7] $\vec{E} + \vec{v} \times \vec{B} = 0$, where $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields, respectively, in the laboratory frame. By linearizing Faraday's law, we obtain

$$B_{1x} = -\frac{1}{\omega}\left(\vec{k}\cdot\vec{B}_0\right)v_{1x}, \tag{5}$$

where we have put $\gamma = -\imath\omega$, since the static state of equilibrium of the system is obviously stable with respect to the perturbation, because dissipative effects are fully absent.

If the direction of $\vec{B}_0$ were constant, the direction of $\vec{k}$ could be always chosen such that $\vec{k}\cdot\vec{B}_0 = 0$ and no bending of the magnetic field lines would be produced. However, due to the shear exhibited by the $\vec{B}_0$-lines, the condition $\vec{k}\cdot\vec{B}_0 = 0$ can be satisfied only at some point in the plasma. The condition $\vec{k}\cdot\vec{B}_0 = 0$ defines the so-called rational surface and, in the neighborhood of the singular point, the distortion of the magnetic field lines provokes the appearance of a restoring force, which exactly opposes to the perturbation-driving force. This is the well-known stabilizing effect due to shear of the Rayleigh-Taylor instability [8].

4. Lorentz force

If we consider a resistive, singularly ionized, approximately neutral plasma, the flux freezing condition may be replaced by Ohm's law in the form [4, 5, 9]

$$\vec{E} + \vec{v}\times\vec{B} = \eta\vec{\jmath} + \mu_0\delta_e^2\frac{\partial\vec{\jmath}}{\partial t}, \tag{6}$$

where η is the plasma resistivity, $\vec{\jmath}$, the current density, μ_0, the vacuum magnetic permeability, and

$$\delta_e = \sqrt{\frac{m_e}{\mu_0 n_e e^2}}, \tag{7}$$

the finite electron skin depth, with m_e, n_e, and e denoting the electron mass, number density, and charge, respectively.

By linearizing Eq. (6), we get

$$\vec{E}_1 + \vec{v}_1\times\vec{B}_{0k} = \left(\eta + \gamma\mu_0\delta_e^2\right)\vec{\jmath}_1, \tag{8}$$

where $\vec{B}_{0k}$ is the component of $\vec{B}_0$ parallel to $\vec{k}$.

Next, by taking the vector product of Eq. (8) with $\vec{B}_{0k}$, we have

$$\vec{E}_1\times\vec{B}_{0k} + \left(\vec{v}_1\times\vec{B}_{0k}\right)\times\vec{B}_{0k} = \left(\eta + \gamma\mu_0\delta_e^2\right)\vec{\jmath}_1\times\vec{B}_{0k}, \tag{9}$$

with the double vector product

$$\left(\vec{v}_1\times\vec{B}_{0k}\right)\times\vec{B}_{0k} = \left(\vec{v}_1\cdot\vec{B}_{0k}\right)\vec{B}_{0k} - \vec{v}_1 B_{0k}^2. \tag{10}$$

The drift term $\vec{E}_1\times\vec{B}_{0k}$, on the lhs of Eq. (9), can be expressed from the linearization of the Faraday and Ampère laws, (for details, see Ref. [4])

$$\imath\hat{x}\times\vec{B}_{0k}\left(\vec{k}\cdot\vec{E}_1'\right) + k^2\vec{E}_1\times\vec{B}_{0k} = -\gamma\mu_0\vec{\jmath}_1\times\vec{B}_{0k}, \tag{11}$$

where the prime denotes a total derivative wrt x.

Finally, by eliminating $\vec{E}_1\times\vec{B}_{0k}$ between Eqs. (9) and (11), and, in the sequel, taking the x-component of the result, we obtain the Lorentz force

$$\hat{x}\cdot\left(\vec{\jmath}_1\times\vec{B}_{0k}\right) = -\left[1 + \left(1 + k^2\delta_e^2\right)\frac{\gamma\tau_D}{k^2 a^2}\right]^{-1}\frac{B_{0k}^2}{\eta}v_{1x}, \tag{12}$$

where we have introduced the diffusion time scale

$$\tau_D = \frac{\mu_0 a^2}{\eta}. \tag{13}$$

5. Viscous force

The x-component of the viscous force is given by

$$\hat{x}\cdot\nabla^2\vec{v}_1 = v_{1x}'' - k^2 v_{1x}. \tag{14}$$

In the neighborhood of the rational surface, the magnetic force on charged species cannot depend on the x-coordinate, otherwise the gravitational field cannot remain constant,

$$\left(v_{1x}B_{0k}\right)' = 0. \tag{15}$$

Close to the singular point, we Taylor-approximate B_{0k} by

$$B_{0k} \sim waB_{0k}', \tag{16}$$

where $w \ll 1$ is a positive, dimensionless number, which determines the width wa of a (viscous-resistive) boundary-layer.

By combining Eqs. (14), (15), and (16), we obtain

$$\hat{x}\cdot\nabla^2\vec{v}_1 = \left(\frac{3}{w^2 a^2} - k^2\right)v_{1x}. \tag{17}$$

6. Viscous-resistive boundary-layer

According to Eqs. (12) and (17), the restoring force is completely determined by the Lorentz + viscous forces, $\hat{x}\cdot\left(\vec{\jmath}_1\times\vec{B}_{0k} + \nu\rho_0\nabla^2\vec{v}_1\right)$, where ν is the fluid kinematic viscosity and ρ_0, the equilibrium mass density. In the neighborhood of the rational surface, the restoring force approaches the gravitational (perturbation-driving) force $\rho_1 g_x$, where ρ_1 is the perturbative mass density. By linearizing the continuity equation, we obtain (recall that $\nabla\cdot\vec{v}_1 = 0$)

$$\rho_1 = -\frac{\rho_0'}{\gamma}v_{1x}. \tag{18}$$

Therefore, by combining Eqs. (12), (17), and (18), we find how to estimate the width wa of the viscous-resistive boundary-layer,

$$\frac{w^4}{w_\eta^4} - \left(\frac{3 - k^2 a^2 w_\nu^2}{3}\right)\frac{w^2}{w_\eta^2} - \frac{w_\nu^2}{w_\eta^2} = 0, \tag{19}$$

where we have introduced the width $w_\eta a$ of a resistive boundary-layer,

$$w_\eta = \sqrt{\left[1 + (1 + k^2\delta_e^2)\frac{\gamma\tau_D}{k^2 a^2}\right]\frac{\kappa\alpha}{\gamma\tau_D}}, \qquad (20)$$

and the width $w_\nu a$ of a viscous boundary-layer,

$$w_\nu = \sqrt{\frac{\gamma\tau_D}{\kappa\alpha}\frac{\nu}{\nu_M}}, \qquad (21)$$

with ν_M denoting a (magnetic) viscosity scale, [10]

$$\nu_M = \frac{v_A^2 \tau_D}{3}, \qquad (22)$$

defined in terms of the Alfvén speed v_A, in the form

$$v_A = \frac{aB_{0k}'}{\sqrt{\mu_0\rho_0}}. \qquad (23)$$

In accordance with Eq. (23), we may also define the Alfvén time scale

$$\tau_A = \frac{a}{v_A}, \qquad (24)$$

which, in turn, determines the "height of free-fall"

$$\alpha = \frac{g_x\tau_A^2}{2} \qquad (25)$$

of charged species, in the constant gravitational field g_x, in the Alfvén time interval τ_A.

From Eqs. (20) and (21), we see that the length scale α is always corrected by the coefficient

$$\kappa = \left(\ln\rho_0^2\right)', \qquad (26)$$

which accounts for possible fluid inhomogeneities.

7. Dispersion relation

In the neighborhood of the rational surface, the gravitational work approaches the variation of the fluid kinetic energy,

$$v_{1x}\rho_1 g_x \sim \gamma\rho_0\left(v_{1x}^2 + v_{1k}^2\right), \qquad (27)$$

where v_{1k} is the component of $\vec{v}_1$ parallel to $\vec{k}$.

The lhs of Eq. (27) can be trivially determined from Eq. (18). To determine the rhs of Eq. (27), first we note that

$$\nabla\cdot\vec{v}_1 = v_{1x}' + \imath k v_{1k} = 0. \qquad (28)$$

Next, from Eqs. (15) and (16), we observe that

$$v_{1x}' \sim -\frac{v_{1x}}{wa}. \qquad (29)$$

By combining Eqs. (28) and (29), we write

$$v_{1k} \sim -\imath\frac{v_{1x}}{wka}. \qquad (30)$$

Since $w \ll 1$, we conclude that

$$v_{1x}^2 + v_{1k}^2 \sim -\frac{v_{1x}^2}{w^2 k^2 a^2}. \qquad (31)$$

Plugging Eqs. (18) and (31) in Eq. (27), we get

$$\frac{\rho_0' g_x}{\gamma} \sim \frac{\gamma\rho_0}{w^2 k^2 a^2}. \qquad (32)$$

Finally, by combining Eqs. (24), (25), and (26) with Eq. (32), we find the dispersion relation of the Rayleigh-Taylor instability

$$\gamma^2\tau_A^2 = \kappa\alpha w^2 k^2 a^2. \qquad (33)$$

From Eq. (33), it is evident that the condition $\kappa\alpha > 0$ must be satisfied for that $\gamma > 0$. In other words, the charged species must "fall down" in the constant gravitational field g_x for that the static state of equilibrium of the system becomes unstable to the linear perturbation.

8. Scaling laws

Quite interestingly, a detailed inspection of Eqs. (9) and (11) reveals that

$$\frac{\gamma\tau_D}{k^2 a^2} \sim \frac{|\vec{E}_1 \times \vec{B}_{0k}|}{|(\vec{v}_1 \times \vec{B}_{0k}) \times \vec{B}_{0k}|}. \qquad (34)$$

When $\gamma\tau_D \ll k^2 a^2$, the contribution of the drift term $\vec{E}_1 \times \vec{B}_{0k}$ to the Lorentz force is negligible and the electromotive force $(\vec{v}_1 \times \vec{B}_{0k}) \times \vec{B}_{0k}$ is sufficient to balance out the dissipative effects, due to the plasma resistivity and fluid viscosity. However, when $\gamma\tau_D \gg k^2 a^2$, the contribution of $(\vec{v}_1 \times \vec{B}_{0k}) \times \vec{B}_{0k}$ to the Lorentz force becomes negligible and $\vec{E}_1 \times \vec{B}_{0k}$ is sufficient to balance-out the dissipative effects.

In the approximation $\gamma\tau_D \ll k^2 a^2$, we have two limiting situations. First, when viscous effects are negligible, $w_\nu \ll w_\eta$, Eq. (19) shows that

$$w \sim w_\eta, \qquad (35)$$

and Eq. (33) furnishes the scaling law

$$\gamma \sim \eta^{1/3}, \qquad (36)$$

which depends on the plasma resistivity and recovers the classical result of the Rayleigh-Taylor instability, as observed in magnetically confined plasmas [8].

Somewhat surprisingly, when resistive effects are negligible, $w_\nu \gg w_\eta$, Eq. (19) shows that

$$w \sim \sqrt{w_\eta w_\nu}, \qquad (37)$$

and Eq. (33) furnishes the scaling law

$$\gamma \sim (\eta\nu)^{1/4}, \qquad (38)$$

which depends on both the plasma resistivity and fluid viscosity.

On the assumption of sufficiently short perturbative wavelengths, $k^2\delta_e^2 \gg 1$, the opposite approximation, $\gamma\tau_D \gg k^2 a^2$, also leads up to two limiting situations. First, when condition (35) holds, Eq. (33) furnishes the scaling law

$$\gamma \sim n_e^{-1/2}, \qquad (39)$$

which depends on the electron number density.

Finally, when condition (37) holds, Eq. (33) furnishes the scaling law

$$\gamma \sim \left(\frac{\nu}{n_{\mathrm{e}}} \right)^{1/3}, \qquad (40)$$

which depends on both the electron number density and fluid viscosity.

9. Conclusion

In this work, the Rayleigh-Taylor instability has been addressed in a viscous-resistive current slab, by assuming a finite electron skin depth. The formulation has been developed on the basis of an extended form of Ohm's law, which includes a term proportional to the explicit time derivative of the current density. In the neighborhood of the rational surface, a viscous-resistive boundary-layer has been defined in terms of a resistive and a viscous boundary-layers. As expected, when viscous effects are negligible, it has been shown that the viscous-resistive boundary-layer is given by the resistive boundary-layer. However, when viscous effects become important, it has been found that the viscous-resistive boundary-layer is given by the geometric mean of the resistive and viscous boundary-layers. Scaling laws of the time growth rate of the Rayleigh-Taylor instability with the plasma resistivity, fluid viscosity, and electron number density have been discussed.

Further developments of our formulation may be of interest for the investigation of plasma instabilities due to sufficiently short perturbative wavelengths. For instance, recently, hydromagnetic shock waves [11], followed by pulsed emissions with wavelengths of the order of 650 nm [12], have been observed. More recently, it has been noted that repeated loading of transient events, like edge localized modes (ELMs), constitute a potential damage to reactor-relevant materials [13]. Actually, it has long been argued that ELMs can yield an intense transient flux of both energy and particles into the ITER divertor [14]. As a matter of fact, during an experiment conducted in a quasi-stationary plasma accelerator (QSPA), at the TRINITI Institute, a significant erosion of tungsten targets has been detected, following their exposition to a series of repetitive pulses, characteristic of type I ELMs [15]. Further refined experiments, as well as numerical simulations, apparently, have confirmed such findings [16].

Acknowledgements

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